NPS55TY74031

NAVAL POSTGRADUATE SCHOOL

Monterey, California

ALLOCATION OF IMPERFECT WEAPONS
FOR ATTACK AND DEFENSE OF
POINT TARGETS

J. B. TYSVER

March 1974

Approved for public release; distribution unlimited

Prepared for:
Special Project Office
Crystal Mall
Washington, D. C. 20390

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

Rear Admiral Masson Freeman
Superintendent

Jack R. Borsting
Provost

The work reported herein was supported by the Special Project Office Crystal Mall, Washington, D.C. 20390.

This report was prepared by:

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NPS55TY74031	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ALLOCATION OF IMPERFECT WEAPONS FOR ATTACK AND DEFENSE OF POINT TARGETS		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) J. B. Tysver		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, California 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS SSP013; TA 82415; WR-4-5022
11. CONTROLLING OFFICE NAME AND ADDRESS Special Project Office, Crystal Mall Washington, D. C. 20390		12. REPORT DATE March 1974
		13. NUMBER OF PAGES 62
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Weapon Allocation, Imperfect Weapons, Point Targets

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This study investigates the effect of imperfect attack and defense weapons on damage estimates and preferred strategies for a set of point targets. The damage estimation model used incorporates weapon success probabilities for both defense and attack weapons. The effect of these probabilities on expected target damage and, subsequently, on weapon allocation preference by both opponents are examined. A Proportional Defense Model is used for illustration and some modification to the model proposed.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601 |

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

I. INTRODUCTION.

In any conflict it is essential that the Offense assess the ability of the Defense to counter an attack. The Defense capabilities depend upon the quality as well as the quantity of defensive weapons. The study discussed in this report was intended primarily to provide information to the Offense of the effect of imperfect Defense weapons on damage estimates and Offense strategies for point targets. It was motivated by concern for target damage predictions in strategic warfare where the defensive weapons are ABM interceptors and the offensive weapons are Re-Entry vehicles.

Evaluation of strategies for the Offense of necessity includes consideration of strategies for the Defense. Further, imperfect offensive weapons are also considered so that a more complete representation of a conflict can be given. Since offensive strategies can only be evaluated against specified defensive strategies, a Proportional Defense Model ([1] and [2]) is used to illustrate the effect of imperfect weapons on both damage estimates and preferred weapon assignments.

This report uses general terms in order that the results be applicable to other conflict situations.

II. BASIC ASSUMPTIONS.

The three elements to be considered in the model used in this study are the offensive weapons (attackers), the defensive weapons (defenders), and the entities which the Offense attempts to destroy and the Defense tries to protect (targets). The model also includes attacker and defender capabilities (success probabilities) so that interaction between Offense and Defense can be evaluated and optimum strategies determined for each side in a conflict.

It is assumed that, when attacked by an offensive weapon, a target either is destroyed or is undamaged (i.e., point targets). Since strategies of the Offense and Defense are to be evaluated and these strategies essentially consist of weapon allocations, a complex of targets T in number will be considered. It will also be assumed that the i^{th} target, T_i, has a value v_i.

The Offense's weapons are assumed to be limited to m in number with each attacker having the same probability r of destroying a target of any value if it is not intercepted by a defensive weapon. The attacker success probability includes target susceptibility as well as weapon reliability, accuracy, and lethal range. When more than one attacker is assigned to a single target, no interference between attackers is considered so that statistical independence can be assumed. The Offense's strategy consists of allocation of its attackers to specific targets thus determining the vector

$$Y = (y_1, y_2, \ldots, y_T)$$

where y_i is the number of attackers allocated to T_i and

$$\sum_{1}^{T} y_i = m.$$

Similarly, the Defense's weapons are assumed to be n in number and to have, individually, the probability p of intercepting (negating) an attack by any single offensive weapon. It is also assumed that there is no interference when they are attempting to intercept attackers on a common target or even when several are assigned to a single attacker. It is assumed that the defensive weapons must be assigned to protect specific targets and are not available for use against offensive weapons attacking other targets (point defense). Defender allocation is described by the vector

$$X = (x_1, x_2, \ldots, x_T)$$

where x_i is the number of defenders assigned to T_i and

$$\sum_{i=1}^{T} x_i = n.$$

The Defense's strategy includes specification of X but must also specify assignment of the defenders against the attackers at any target.

The above assumptions provide the basis for a relatively simple model which can be used to obtain some insight of a conflict situation and which can be used as a basic building block for more realistic models. The assumptions are quite restrictive and would require considerable modification to make them descriptive of an actual conflict.

III. THE MODEL.

The expected survival value of a target when it has j defenders and k attackers will be denoted by $S(j,k)$. When an undefended target T_i is attacked by a single offensive weapon, the probability that the target survives the attack is $(1-r)$ and the expected value of the target after the attack is

$$S_i(0,1) = v_i(1-r).$$

If the target has a single defender, its expected survival value is

$$S_i(1,1) = v_i(1-rq)$$

where $q = 1-p$ is the probability that the defender does not successfully intercept the attacker. If T_i has x_i defenders and one attacker and all of the defenders are used against the attacker, then the survival value is

$$S_i(x_i,1) = v_i(1-rq^{x_i}).$$

If, on the other hand, T_i has one defender and y_i attackers, then its survival value is

$$S_i(1,y_i) = v_i(1-rq)(1-r)^{y_i-1}.$$

The survival value when T_i has x_i defenders and y_i attackers depends not only on the magnitudes of x_i and y_i but also upon how the defenders are assigned. If, for example, x_i is less than y_i and each defender is assigned to a different attacker (1-on-1) defense then

$$S_i(x_i,y_i) = v_i(1-rq)^{x_i}(1-r)^{y_i - x_i}$$

on the other hand, if x_i is greater than y_i the Defense can assign more than one defender against at least one of the attackers. Let k denote the greatest integral multiple of y_i in x_i, $(k = [\frac{x_i}{y_i}])$, then

$$x_i = ky_i + j$$

where k and j are integers and j is less than y_i. If the Defense distributes all of its defenders as evenly as possible against the attackers, then the survival value of the target is

$$S_i(x_i y_i) = v_i(1-rq^{k+1})^j (1-rq^k)^{y_i-j}.$$

Let a given complex of targets $(T_i;\ i=1,2,\ldots,T)$ with their associated values $(v_i;\ i=1,2,\ldots,T)$ have n defenders assigned according to the Defense's assignment vector X. Where the Offense uses m weapons distributed according to its assignment vector Y, then the total expected survival value of the target complex is

$$S(X,Y) = \sum_1^T S_i(x_i,y_i).$$

From the point of view of the Offense the expected damage to the target complex is

$$D(X,Y) = V - \sum_1^T S_i(x_i,y_i) = V - S(X,Y)$$

$$= \sum_1^T [v_i - v_i(x_i,y_i)]. \equiv \sum_1^T D_i(x_i,y_i)$$

where $D_i(x_i,y_i)$ is the expected damage to target T_i and $V = \sum_1^T v_i$. It will be convenient in parts of the later development to express the expected survival and damage values of T_i as the products

$$S_i(x_i,y_i) = v_i P(x_i,y_i)$$

and

$$D_i(x_i,y_i) = v_i[1-P(x_i,y_i)]$$

where $P(x_i,y_i)$ is the probability that a point target survives when it has x_i defenders and y_i attackers. In particular, when x_i is less than y_i and 1-on-1 defense is used,

$$P(x_i,y_i) = (1-rq)^{x_i}(1-r)^{y_i-x_i}.$$

IV. THE PROBLEM OF WEAPON ALLOCATION.

In allocating its n defensive weapons to the targets, the Defense tries to maximize the total survival value $S(X,Y)$ by selection of the vector X. Similarly, the Offense tries to maximize the expected total damage value $D(X,Y)$ by selection of the vector Y. Maximization of $D(X,Y) = V - S(X,Y)$ is equivalent to minimization of $S(X,Y)$ and the problem can be treated as a Zero Sum Two-Person Game where the two opponents (Offense and Defense) select strategies (assignment vectors Y and X, respectively).

The problem is complicated by the fact that the two opponents do not have complete information on the strategies available to each other. One of the important elements of information is the sizes of the opposing forces (m and n). Some effort has been directed to the investigation of fractional allocation where $x = y = 1$ and $x_i y_i$ is the proportion of the Defense (Offense) forces allocated to the target T_i ([1], Chapter 4). Another approach for the Defense is the allocation of defensive weapons so that the target damage per offensive weapon does not exceed a specified value for any attack size or attacker allocation. This Defense allocation is called Proportional Defense ([2]) and is discussed in Section 5 of this report.

A further compounding of the Defense allocation aspect of the problem is temporal. In general, the Defense must pre-allocate its resources whereas the Offense can delay its allocation until the start of the attack (Offense has last move) and hence can take advantage of any information it can obtain on Defense allocation. Further, the Offense can mask the total attack size y_i on target T_i by distributing the attack in

several groups separated in time (waves). If each attack wave on a target contains a single attacker the attack is called sequential. Proportional Defense is designed specifically for sequential attacks.

In addition to its problem of allocating defenders to targets, the Defense must also decide how to distribute its x_i defenders of target T_i among the y_i attackers if the attack on T_i occurs in a single wave. If the attack on T_i does not occur in a single wave, the Defense must decide how many defenders to commit against each wave and how they are to be distributed among the attackers in the wave. The Defense preference for distributing defenders among the attackers in a single wave on a target is treated in Appendix A and is relatively clear-cut. An appropriate modification of Proportional Defense against wave attacks is proposed in Section 6.

V. PROPORTIONAL DEFENSE CONCEPT.

A particular strategy for the Defense which can be used against any strategy for the Offense is called Proportional Defense. It is an appropriate defensive strategy when the Defense has no information on the strategy or stockpile size (number of offensive weapons). It is introduced here both to demonstrate the effect of imperfect weapons when this specific defensive strategy is used, and to provide a reference for the investigation of the effect of imperfect weapons on damage estimation and the strategies of the Offense and Defense in general.

The basic concept of Proportional Defense is defined as follows [1]. Let A denote the attack size which exhausts the stockpile of defenders for a set of point targets and let F denote the expected fraction of targets destroyed by A attackers. The expected fraction of targets destroyed by an attack of size y for $y \geq A$ is denoted by $D(y)$. Under appropriate assumptions ([1]) this can be expressed in the form

$$D(y) = F + (1-F)[1-e^{-a(y-A)}]$$

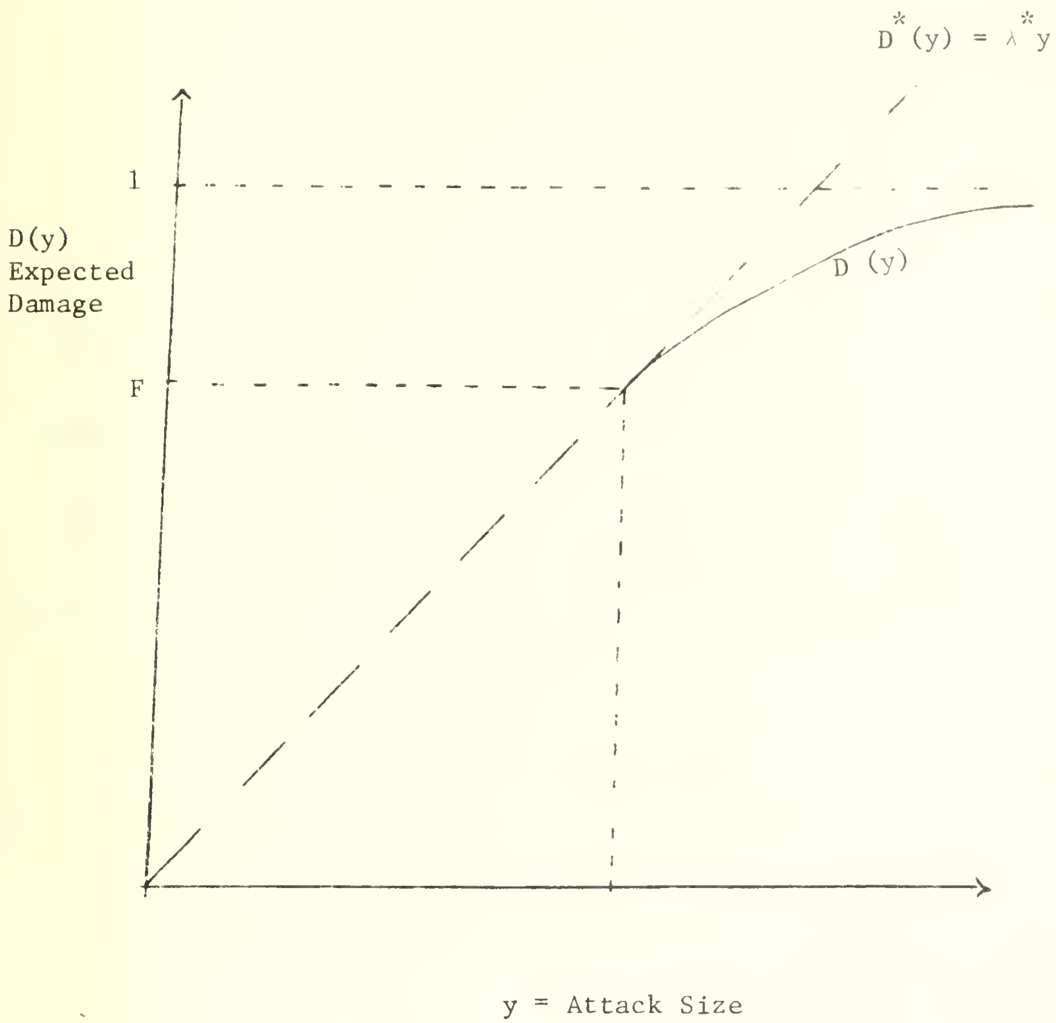
where a is dependent upon target hardness and offensive weapon yield and accuracy. In selecting its strategy the Defense determines a pair (F,A) so that

$$F/A = \max_{y} D(y)/y.$$

For this defensive strategy selection, the fraction of targets destroyed will be essentially proportional to the attack size. Further for each target destroyed, the Offense is forced to pay a price that is proportional

to the value of that target. The payoff to the Offense in terms of expected damage per attacker for any attack of size A or larger will be no greater than $\lambda^* = F/A$.

The principal is illustrated in the following sketch (Figure 1).



Expected Damage for Proportional Defense

FIGURE 1

VI. A PROPORTIONAL DEFENSE MODEL.

The basic principal of Proportional Defense can be incorporated into a Proportional Defense Model which is in a constructive form, [2]. Consider a sequential attack against a single target T_i. A sequential attack of size y_i on T_i is a multiple wave attack consisting of y_i waves with each wave consisting of a single attacker ($y_{ij} = 1$ for the j^{th} wave where $j = 1,2,\ldots,y_i$). The defense of T_i must be such that for any attack size y_i the payoff of the Offense in terms of expected target value destroyed per attack weapon is no greater than a value d^* which is specified by the Defense. The defensive strategy is to be such that

$$\frac{D_i(x_i,y_i)}{y_i} \leq d^*$$

for all y_i. The <u>minimum number</u> of defenders, x_i, to be assigned in defense of T_i and the appropriate assignments of defenders against each attack wave (individual attackers) must be established. A Proportional Defense for T_i is described below. The procedure is sequentially constructive.

Consider the first attack wave ($y_{i1} = 1$). The number of defenders, x_{i1}, to be assigned to the first attacker must be such that the expected damage to T_i is less than d^*. This expected damage is

$$D_{i,1}(x_{i1},1) = v_i[1-(1-rq^{x_{i1}})] = v_i rq^{x_{i1}}.$$

Thus x_{i1} is selected as the smallest integer such that

$$v_i rq^{x_{i1}} \leq d^*.$$

At each succeeding attack wave, the Defense considers the possibility of reducing the number of defenders x_{ik}, without violating the payoff restriction on that wave or on subsequent waves. Thus, if

$$\sum_{j=1}^{\ell} D_{i,j}(x_{ij},1) \le \ell d^*$$

with $x_{ij} = x_{i,k-1}-1$ when $j \ge k$ for all $\ell \ge k$, then $x_{ik} = x_{i,k-1}-1$. Otherwise $x_{ik} = x_{i,k-1}$. The construction of the defensive weapon assignments is illustrated in the following sketch (Figure 2), when $x_{i1} = 3$. In this sketch the expected damage to the target as a function of $y_i = j$ is

$$D_i(x_i,j) = \begin{cases} v_i[1=(1-rq^3)^j] & \text{for } j \le j_3 \\ D_i(x_{i(3)},j_3) + [v_i-D_i(x_{i(3)},j_3)][1-(1-rq^2)^{j-j_3}] & \text{for } j_3 < j \le j_2 \\ D_i(x_{i(2)},j_2) - [v_i-D_i(x_{i(2)},j_2)][1-(1-rq)^{j-j_2}] & \text{for } j_2 < j \le j_1 \\ D_i(x_i,j_1) - [v_i-D_i(x_i,j_1)][1-(1-r)^{j-j_1}] & \text{for } j > j_1. \end{cases}$$

where

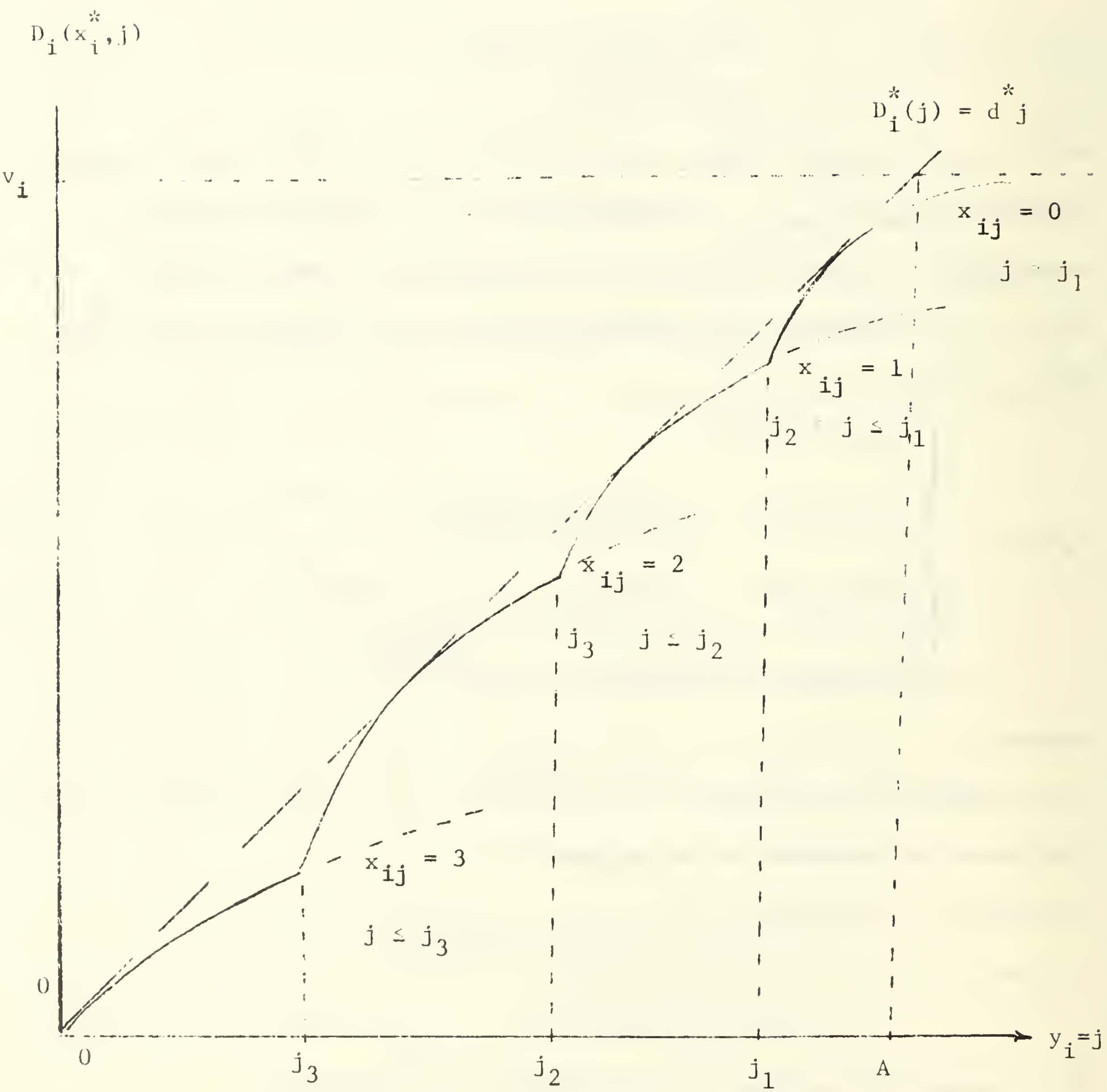
$$x_{i(3)} = 3j_3, \quad x_{i(2)} = 3j_3 + 2(j_2-j_3), \quad x_i = 3j_3 + 2(j_2-j_3) + (j_1-j_2)$$

The number of defenders to be assigned to T_i is

$$\begin{aligned} x_i &= 3j_3 + 2(j_2-j_3) + (j_1-j_2) \\ &= j_3 + j_2 + j_1. \end{aligned}$$

For an attack of any size y_i the payoff to the Offense is

$$\frac{D_i(x_i,y_i)}{y_i} \le d^*.$$



Construction of Defender Assignments for T_i

FIGURE 2

The defensive weapon supply is exhausted when j_1 offensive weapons have been used.

The average expected payoff per attacker for T_i is denoted by $d_i(y_i)$, i.e.,

$$d_i(y_i) = D_i(x_i,y_i)/y_i.$$

When the Offense uses an attack of size $y = \sum_1^T y_i$ against a set of targets $(T_1,T_2,\ldots,T_T)$, the total expected damage is

$$D(X,Y) = \sum_{i=1}^{T} D_i(x_i,y_i)$$

and the average expected damage per attacker is

$$d = D(X,Y)/y = \frac{1}{y}\sum D_i(x_i,y_i).$$

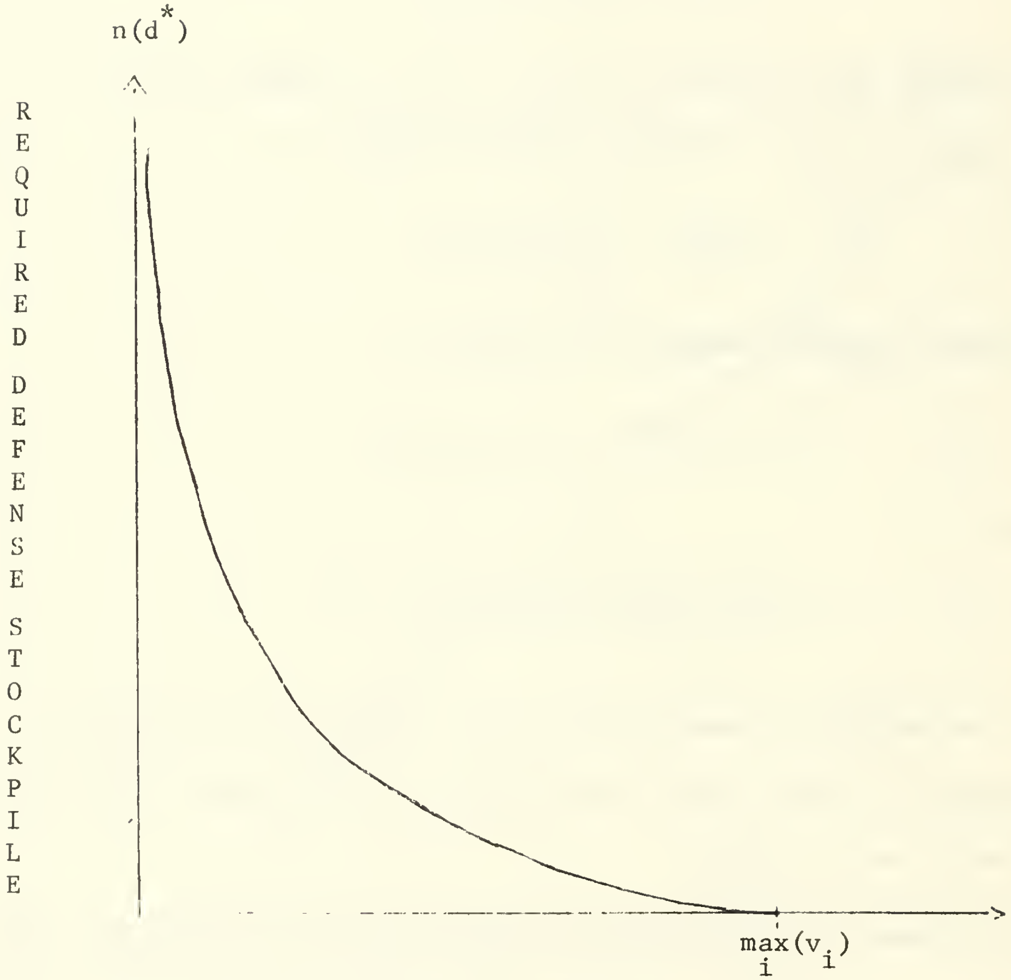
Note that

$$d = \frac{1}{y}\sum_1^T y_i d(y_i) \leq \frac{1}{y}\sum y_i d^* = d^*$$

so that a Proportional Defense at each target which limits the payoff to the Offense for that target also provides the same payoff limitation for the entire set of targets.

Establishment of a Proportional Defense for a set of unequal valued point targets is simple in principle although the procedure may be lengthy and involved. Given the values of r, p, and the v_i's, the number of defenders n and their allocation vector X can be determined for any payoff limitation d^*. Note that, in particular, n is a function of d^* as shown in the accompanying figure (Figure 3).

On the other hand, for a given value n the corresponding payoff



d^* = payoff limitation (damage per attacker)

Proportional Defense Stockpile Requirement

FIGURE 3

and the appropriate defense allocation vector are not so readily determined. For example, if the defenders were assigned to targets in proportion to the target values (i.e., $x_i = \frac{v_i}{\sum v_i} n$) unequal payoffs could occur for the individual target attacks. Instead the Defense must consider the total defender stockpile n as a function of d^* and estimate d^* as an inverse function of n. To do this the Defense can determine the number of defenders $n(d^*)$ required for specific values of d^* and construct the curve representing this functional relationship. For any specific value of n the appropriate value of d^* can be estimated. The selected value of d^* can then be used to determine the defender allocation vector X.

VII. APPLICATION OF PROPORTIONAL DEFENSE.

Theoretical investigations commonly assume that the maximum payoff, d^*, is actually achieved against a Proportional Defense by any attack size. This would be true only if the defenders at each target were not restricted to integers. A few examples will be given to illustrate this aspect of Proportional Defense as well as some other problems of this strategy.

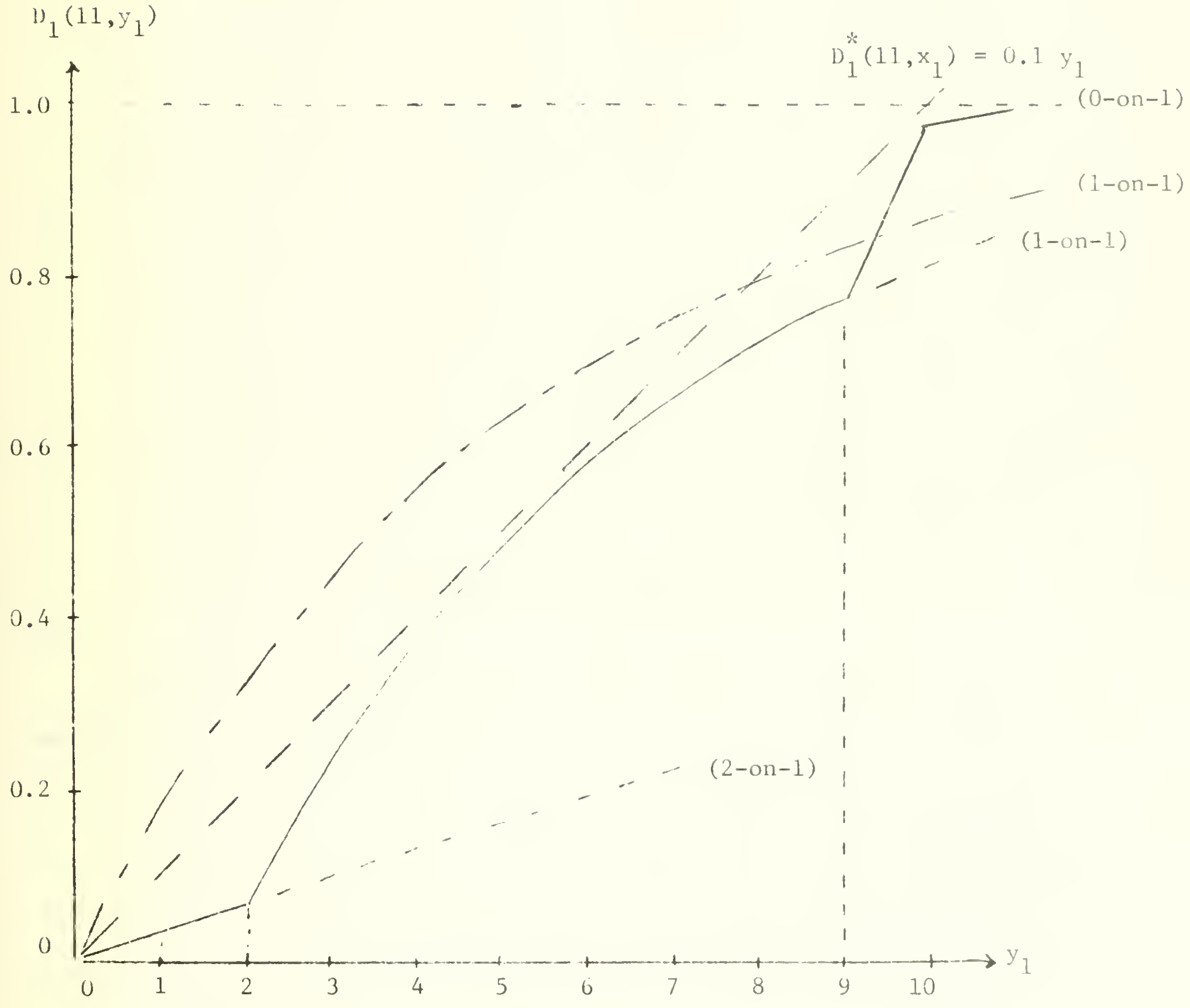
For the first example consider a single target, T_1, with target value $V_1 = 1$. Let $r = 0.9$ and $p = 0.8$. A Proportional Defense for T_1 which limits the payoff to the Offense at $d^* = 0.1$ requires $x_1 = 11$ defenders to produce the following estimated damage as a function of the number of attackers y_1 assigned by the Offense to this target.

$$D_1(11,y_1) = \begin{cases} 1 - (1-rq^2)^{y_1} = 1 - (.964)^{y_1} & 0 \leq y_1 \leq 2 \\ 1 - (1-rq^2)^2(1-rq)^{y_1-2} = 1 - (.9293)(.82)^{y_1-2} & 2 < y_1 \leq 9 \\ 1 - (1-rq^2)^2(1-rq)^7(1-r)^{y_1-9} = 1 - (.2723)(.1)^{y_1-9} & 9 < y_1 \end{cases}$$

This damage function is shown in the accompanying figure (Figure 4). Two defenders are assigned against each of the first two attackers and one defender against each of the next seven attackers. The defense of T_1 is then exhausted.

Three details of this defense are worthy of special comment. First, if the Offense uses attack size $x_1 = 5$ or $x_1 = 10$ the average expected damage per attacker (payoff) is $d = 0.0976$ or 0.0977, respectively. On the other hand, if the Offense uses attack size $x_1 = 2$ or $x_1 = 9$ the

$v_1 = 1 \quad r = 0.9 \quad p = 0.8 \quad d^* = 0.1$



Proportional Defense of T_1

FIGURE 4

payoff is lower (0.0353 or 0.0854, respectively). This consequence of integer defense increments should be considered by the Offense in selecting its attack size.

Second, if the Defense had information that the attack on T_1 consisted of more than nine attackers, then a 1-on-1 defense (broken line) would be better than a Proportional Defense for T_1. The third comment pertains to the expected damage by the tenth attacker of T_1 which is $(0.9768 - 0.7683) = 0.2083$ and is greater than d^*. Suppose there is another target T_2 with value V_2 such that $0.108 \leq V_2 \leq 0.2178$, then a single defender would be allocated to T_2 under Proportional Defense. If no defender were assigned to T_2, the expected damage to T_2 by a single attacker would be

$$D_2(1,1) = V_2[1-(1-r)] \leq 0.19602.$$

Less total expected damage to the two targets would be incurred when T_1 is attacked by ten attackers then if the defender for T_2 were shifted to T_1. These two comments demonstrate that optimality, in the sense of minimizing expected target damage, is not necessarily achieved by Proportional Defense either for a specific target or for a set of targets with different values.

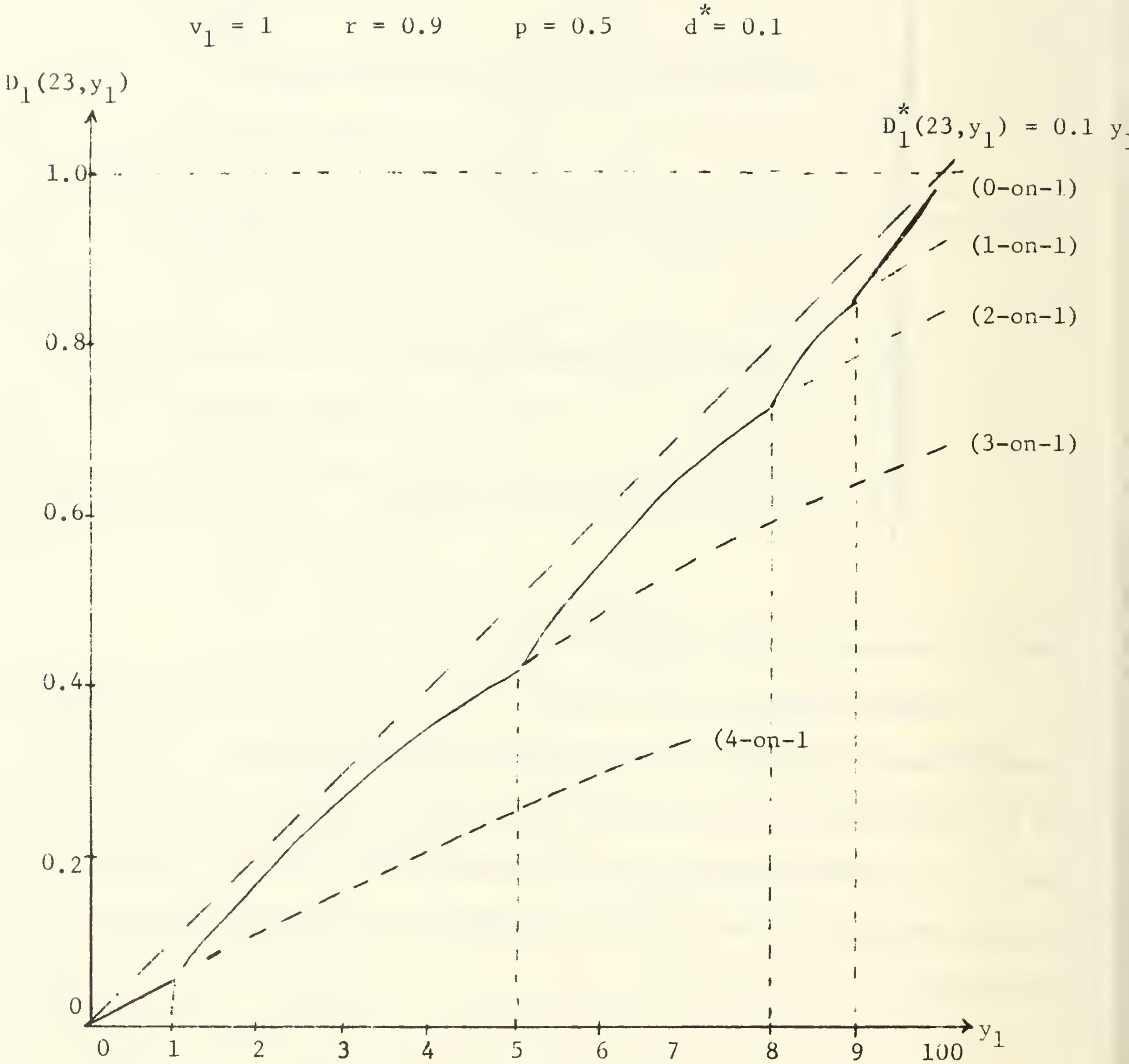
The second example is introduced to demonstrate that a target defense with smaller increments of contribution by defenders can approach its continuum bound approximation $D_i^*(x_i,y_i) = d^*y_i$. Hence, it will be less advantageous for the Offense to attempt to select attack sizes with payoffs close to d^* and to avoid attack sizes with payoffs substantially below d^*. This point is illustrated by reducing the defender success probability

in the first example to $p = 0.5$. The expected damage to T_1 is

$$D_1(23,y_1)=\begin{cases} 1-(1-rq^4)^{y_1} = 1-(0.94375)^{y_1} & \text{for } 0 \le y_1 \le 1 \\ 1-(1-rq^4)(1-rq^3)^{y_1-1} = 1-(0.94375)(.8875)^{y_1-1} & \text{for } 1 < y_1 \le 5 \\ 1-(1-rq^4)(1-rq^3)^4(1-rq^2)^{y_1-5} = 1-(0.5855)(0.775)^{y_1-5} & \text{for } 5 < y_1 \le 8 \\ 1-(1-rq^4)(1-rq^3)^4(1-rq^2)^3(1-rq)^{y_1-8} = 1-(0.2725)(0.55)^{y_1-8} & \text{for } 8 < y_1 \le 9 \\ 1-(1-rq^4)(1-rq^3)^4(1-rq^2)^3(1-rq)(1-r)^{y_1-9} = 1-(0.1499)(.1)^{y_1-9} & \text{for } 9 < y_1 \end{cases}$$

and is presented graphically in Figure 5.

The third example is selected to demonstrate the effect of changing target value or payoff. Let the success probabilities be $r = 0.9$ and $p = 0.8$ as in the first example. The value of target T_3 is also set at $v_3 = 1$ but the payoff limit is reduced to $d^* = 0.01$. A Proportional Defense requires $x_3 = 236$ defenders. The expected damage function is



Proportional Defense of T_1

FIGURE 5

$$D_3(236,y_3)=\begin{cases} 1-(1-rq^3)^{y_3} = 1-(0.9928)^{y_3} & 0 \le y_3 \le 47 \\ 1-(1-rq^3)^{47}(1-rq^2)^{y_3-47} = 1-(0.7119)(0.964)^{y_3-47} & 47 < y_3 \le 90 \\ 1-(1-rq^3)^{47}(1-rq^2)^{43}(1-rq)^{y_3-90} = 1-(0.1472)(0.82)^{y_3-90} & 90 < y_3 \le 99 \\ 1-(1-rq^3)^{47}(1-rq^2)^{43}(1-rq^2)^{9}(1-r)^{y_3-99} = 1-(0.0247)(0.1)^{y_3-99} & 99 < y_3 \end{cases}$$

This expected damage is shown in Figure 6.

Proportional Defense requires that

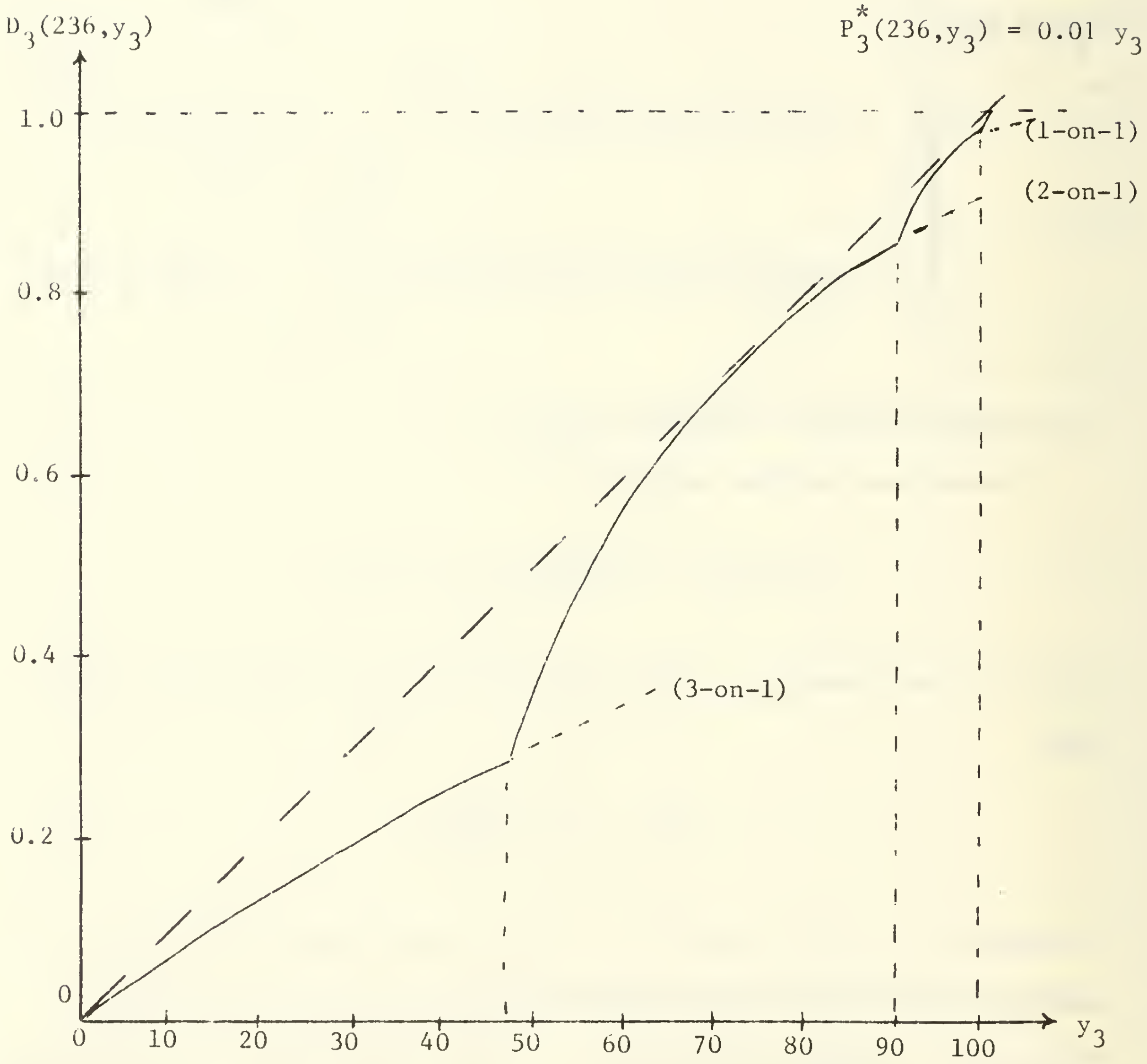
$$D_i(x_i,y_i) = v_i[1-P(x_i,y_i)] \le d^* y_i$$

for all attack sizes y_i. This requirement can be re-expressed in the form

$$1 - P(x_i,y_i) \le \left(\frac{d^*}{v_i}\right)y_i.$$

Note that if $v_3 = 10$ and $d^* = 0.1$ then the Proportional Defense would be identical to the one presented above. Thus a Proportional Defense for a target of any value v_i can be determined by establishing the corresponding defense for a target of unit value with the appropriate adjustment in payoff limit. Given the success probabilities, r and p, the Defense can calculate the required numbers of defenders for a unit

$v_3 = 1$ (or 10) $r = 0.9$ $p = 0.8$ $d^* = 0.01$ (or 0.1)



Proportional Defense of T_3

FIGURE 6

valued target when the payoff limit has specified value. The curve constructed from this data is similar to that in Figure 3 and can be used to estimate the defense requirements x_i for each target T_i in any target set and specified payoff limit d^*. The total number of defensive weapons $n = \sum_1^T x_i$ and associated d^* provide the inputs to be used as in Section VI (see Figure 3).

VIII. PROPORTIONAL DEFENSE AGAINST A WAVE ATTACK.

The Proportional Defense Model designed for use against a sequential attack on a target can be modified to make it applicable against a wave attack. Suppose a wave attack on T_i occurs in K waves with $x_{i(K)}$ attackers in the K^{th} wave with the size and number of subsequent attack waves unknown to the Defense. If any attack wave contains attackers for which only one or two defense levels (number of defenders per attacker) would be committed under Proportional Defense then that defense is appropriate. However, if three or more defense levels would be used, then the Defense's capability can be improved by allocating the same total number of defenders as uniformly as possible against the attackers in the wave. This is in accord with the principle described in Appendix A.

To illustrate this modification consider the situation in the second example with a first attack wave of size $y_{1(1)} = 6$. Proportional Defense would assign four defenders against one of the attackers, three defenders against each of four attackers, and two defenders against the remaining attacker. The expected damage with this defense is

$$D_1(23,6) = 0.5462.$$

If the 18 defenders are divided equally among the attackers with three defenders assigned to each, the expected damage is

$$D_1'(23,6) = 1 - (1-rq^3)^6 = 0.5114.$$

From the Defense's point of view, wave attacks would be preferable to a sequential attack of the same size since there is the possibility of balancing the defense within each wave and thus reducing the expected damage. From the Offense's point of view, any attack on a target should

be distributed in small waves so that the Defense cannot achieve this reduction. It should be recognized, however, that the validity of these strategies for both the Defense and Offense is a consequence of the assumptions used in establishing the model. Further, practical considerations may limit the capability of either side of a conflict from taking advantage of the potentially beneficial modifications.

IX. WEAPON ALLOCATION FOR TWO TARGETS.

The preceding sections of this report dealt primarily with Proportional Defense. Some indication of the effect of imperfect weapons on preferred weapon allocations can be demonstrated by examining a conflict situation involving only two targets. The results established here would be applicable when either opponent had information on the opposition's weapon allocation to the targets.

Consider two targets, T_1 and T_2, of equal value, $v_1 = v_2 = 1$. When the Defense and Offense use weapon allocations $X = (x_1 x_2) = (i,j)$ and $Y = (y_1,y_2) = (k,\ell)$, respectvely, the expected survival values of T_1 and T_2 are $S_1(i,k)$ and $S_2(j,\ell)$. The total survival value is

$$S(X,Y) = S_1(i,k) + S_2(j,\ell).$$

Similarly, the total expected damage is

$$\begin{aligned} D(X,Y) &= D_1(i,k) + D_2(j,\ell) \\ &= 2 - S_1(i,k) - S_2(j,\ell). \end{aligned}$$

The investigation will determine preferred weapon allocations for each opponent given specific allocations by the opposition. When the opposition's allocation is specified, the notation can be simplified by deleting them in the corresponding symbol. Thus, when $Y = (k,\ell)$ the total expected survival value will be denoted by

$$S(i,j) = S_1(i,k) + S_2(j,\ell)$$

to indicate the dependence on the defensive allocation $X = (i,j)$. Similarly, when $X = (i,j)$ the total expected damage is

$$D(k,\ell) = 2 - S_1(i,k) - S_2(j,\ell).$$

to indicate the dependence on the attack allocation $Y = (k,\ell)$ For given $Y = (k,\ell)$, the Defense wants to divide its available number of defenders, n, so as to maximize $S(i,j)$ with $i + j = n$. The Offense, on the other hand, for given $X = (i,j)$ and specified number of attackers, m, wants to maximize $D(k,\ell)$ with $k + \ell = m$.

X. DEFENSE OF TWO TARGETS.

Preferred defensive weapon allocations against several specific attacker allocations are outlined in Appendix B. One of these is described here. Let $Y = (1,2)$ and $n = 3$. The total survival values for different allocations of the defenders are

$$S(3,0) = S_1(3,1) + S_2(0,2) = (1-rq^3) + (1-r)^2,$$

$$S(2,1) = S_1(2,1) + S_2(1,2) = (1-rq^2) + (1-rq)(1-r).$$

$$S(1,2) = S_1(1,1) + S_2(2,2) = (1-rq) + (1-rq)^2, \text{ and}$$

$$S(0,3) = S_1(0,1) + S_2(3,2) = (1-r) + (1-rq^2)(1-rq).$$

Now consider the differences in survival values. Since

$$S(1,2) - S(0,3) = rp[p + r\,(1-p)^2]$$

and is positive for all probabilities r and p, the defender allocation $X = (0,3)$ can be eliminated. The other differences are

$$S(3,0) - S(2,1) = rp(r-2p+p^2) \geq 0 \qquad \text{when } r \geq 2p - p^2 \equiv r^*,$$

$$S(2,1) - S(1,2) = rp(r-rp-p) \geq 0 \qquad \text{when } r \geq \frac{p}{1-p} \equiv r^{**}, \text{ and}$$

$$S(3,0) - S(1,2) = rp(2r(2r-rp+p^2-3p) \geq 0 \text{ when } r \geq \frac{3p-p^2}{2-p} = r^{***}.$$

Note that, when $p = 0.382$ the three conditions have the common value $r^* = r^{**} = r^{***} = 0.618$. If $p \leq 0.382$, the preferred defender allocations are

$$X = \begin{cases} (3,0) & \text{when } r \geq r^{*} \\ (2,1) & \text{when } r^{**} \leq r \leq r^{*} \\ (1,2) & \text{when } r < r^{**}. \end{cases}$$

If $p > 0.382$ the preferred allocations are

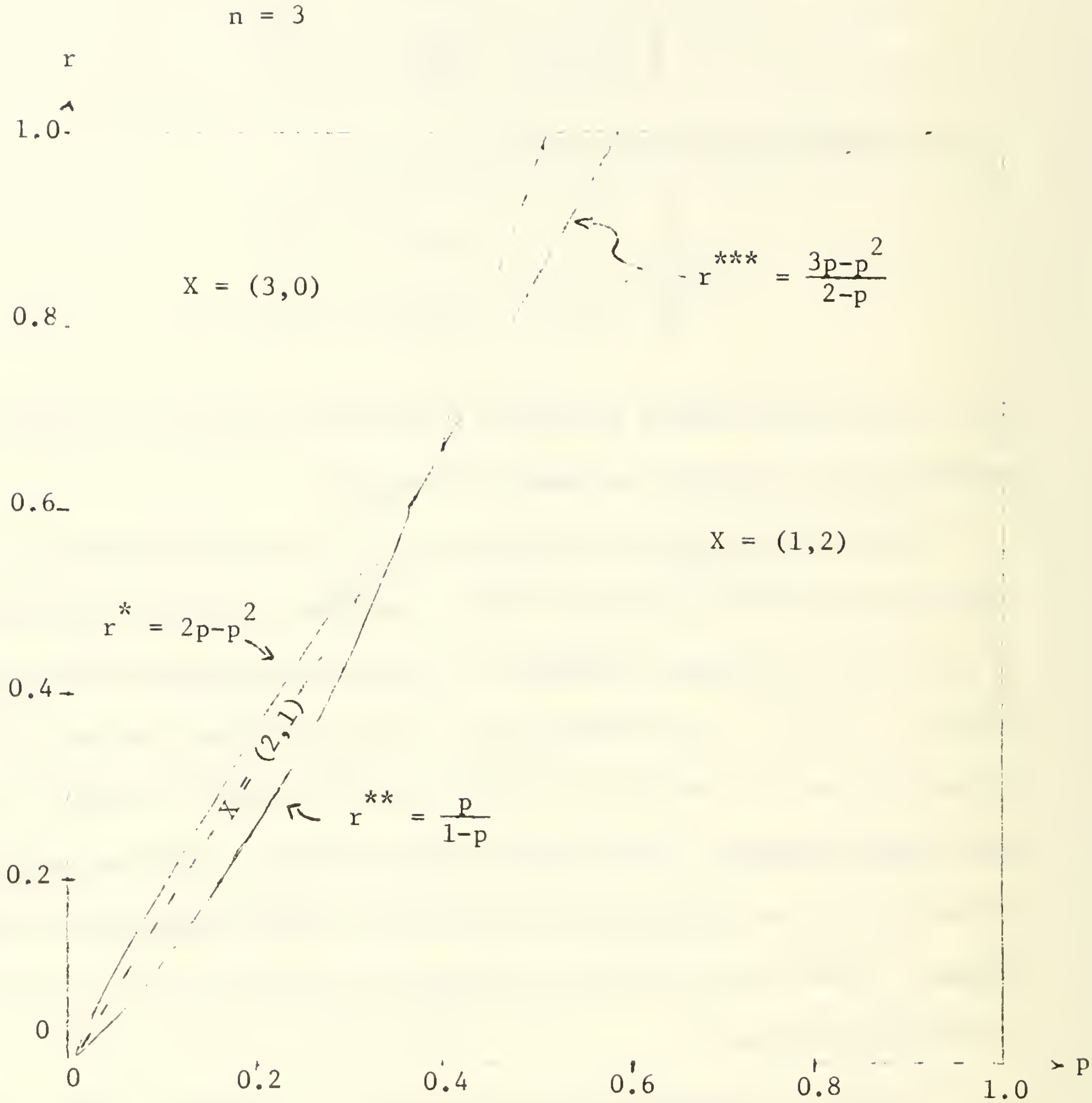
$$X = \begin{cases} (3,0) & \text{when } r \geq r^{***} \\ (1.2) & \text{when } r < r^{***}. \end{cases}$$

The relationship between preferred defense allocation and the success probabilities r and p is shown in Figure 7.

Preferred defensive allocations and the corresponding total survival values are presented in Table 1 for a few limited attack allocations, $Y=(k,\ell)$, and Defense stockpiles, n. Only one attacker success probability, $r = 0.9$, is included but a range of defender success probabilities is used to demonstrate the shift in preferred defensive allocation with this parameter. The table is only partially filled but gives an indication of the effect of p on the allocation preference for the Defense. Some general defensive allocation procedures can be developed as indicated below.

Consider a balanced attack, $Y = (k,k)$ on the two targets. Suppose that the defensive allocation is $X = (i,j)$ with $i \leq j < k$, and an additional defender is to be allocated to one of the targets. If the defender is assigned to T_1, then the expected total survival value is

$$\begin{aligned} S(i+1,j) &= S_1(i+1,k) + S_2(j,k) \\ &= (1-rq)^{i+1}(1-r)^{k-i-1} + (1-rq)^{j}(1-r)^{k-j}. \end{aligned}$$



PREFERRED DEFENSE X AGAINST ATTACK Y = (1,2)

FIGURE 7

TABLE 1

Two-Target Defense Allocation, X(i,j), Against Attack, Y(k,ℓ)

r = 0.9

Offense	Defense	p = 0.3		p = 0.5		p = 0.7		p = 0.9		p = 1.0	
Y = (k,ℓ)	n	Pref.X (i,j)	S(i,j)	Pref. X (i,j)	Surv. S(i,j)	Pref.X (i,j)	Surv. S(i,j)	Pref.X (i,j)	Surv. S(i,j)	Pref.X (i,j)	Surv. S(i,j)
(1,1)	0	(0,0)	0.2		0.2		0.2		0.2		0.2
	1	(1,0)	.47	(1,0)	.65	(1,0)	.83	(1,0)	1.01	(1,0)	1.1
	2	(1,1)	.74	(1,1)	1.10	(1,1)	1.26	(1,1)	1.82	(1,1)	2.0
	3	(2,1)	.929	(2,1)	1.325	(2,1)	1.649	(2,1)	1.901	(2,1)	
	4	(2,2)	1.118	(2,2)	1.550	(2,2)	1.838	(2,2)	1.982		
	5	(3,3)	1.256	(3,2)	1.662	(3,2)	1.894	(3,2)	1.990		
	6	(3,3)									
(1,2)	0	(0,0)	0.11		0.11		0.11		0.11		0.11
	1	(1,0)	.38	(1,0)	.56	(1,0)	.74	(1,0)	.92	(1,0)	1.01
	2	(2,0)	.57	(2,0)	.79	(2,0)	.93	(1,1)	1.00	(1,1)	1.10
	3	(3,0)	.70	(3,0)	.90	(1,2)	1.26	(1,2)	1.74	(1,2)	2.0
	4	(4,0)	.79	(2,2)	1.08	(2,2)	1.45	(2,2)	1.83		
	5										
	6										
(2,2)	0	(0,0)	.02		.02		.02		.02		.02
	1	(1,0)	.05	(1,0)	.07	(1,0)	.08	(1,0)	.10	(1,0)	.11
	2	(2,0)	.147	(2,0)	.312	(2,0)	.407	(2,0)	.838	(2,0)	1.01
	3	(3,0)	.217	(3,0)	.436	(3,0)	.589	(2,1)	.919	(2,1)	1.1
	4	(4,0)	.322	(4,0)	.611	(2,2)	.794	(2,2)	1.656	(2,2)	2.0
	5										
	6										

If the defender is assigned to T_2, the survival value is

$$S(i,j+1) = S_1(i,k) + S_2(j+1,k)$$

$$= (1-rq)^i(1-r)^{k-i} + (1-rq)^{j+1}(1-r)^{k-j-1}.$$

The difference in survival values is

$$\Delta S = S(i,j+1) - S(i+1,j)$$

$$= rp(1-rq)^j(1-r)^{k-j-1}[1-(\frac{1-r}{1-rq})^{j-1}].$$

Since $(1-r) \leq (1-rq)$, this difference is positive for all values of the probabilities r and q. The additional defender should be assigned to the target (T_2 in this case) which already has the largest number of defenders. Thus the first k defenders should all be, assigned to the same target.

Next, suppose that the first k defenders have been assigned to T_2 when the attack is $Y = (k,k)$. To which target should an $(k+1)^{st}$ defender be assigned? If it is assigned to T_1, the survival value is

$$S(1,k) = S_1(1,k) + S_2(k,k)$$

$$= (1-rq)(1-r)^{k-1} + (1-rq)^k$$

and, if it is assigned to T_2,

$$S(0,k+1) = S_1(0,k) + S_2(k+1,k)$$

$$= (1-r)^k + (1-rq^2)(1-rq)^{k-1}.$$

The difference is

$$\Delta S = S(1,k) - S(0,k+1)$$

$$= rp[(1-r)^{k-1} - q(1-rq)^{k-1}]$$

and will be positive when

$$(1-r)^{k-1} \geq q(1-rq)^{k-1}.$$

If $k = 1$, this will be positive for all r and p and the additional defender should be assigned to T_1. If, however, $k > 1$ then the assignment preference will depend upon k as well as r and p.

To generalize this, let T_2 have k defenders and T_1 have i defenders where $0 \leq i < k$, where the attack is $Y = (k,k)$. To which target should an additional defender be assigned? The expected survival values are

$$S(i+1,k) = S_1(i+1,k) + S_2(k,k)$$

$$= (1-rq)^{i+1}(1-r)^{k-i-1} + (1-rq)^k \quad \text{and}$$

$$S(i,k+1) = S_1(i,k) + S_2(k+1,k)$$

$$= (1-rq)^i(1-r)^{k-i} + (1-rq^2)(1-rq)^{k-1}.$$

The difference is

$$\Delta S = S(i+1,k) - S(i,k+1)$$

$$= rp(1-rq)^i[(1-r)^{k-i-1} - q(1-rq)^{k-i-1}]$$

and will be positive when

$$(1-r)^{k-i-1} \geq q(1-rq)^{k-i-1}.$$

If $i = k-1$ this inequality will be satisfied for all k, r, and p and the additional defender should be assigned to T_1 giving both targets k defenders. If, however, $i < k-1$, then the preferred assignment will depend upon $(k-i)$ as well as upon r and p.

The above development of preferred defensive allocation against a balanced attack, $Y = (k,k)$, can be extended to situations in which the stockpile for the Defense is greater than that for the Offense. Consider a defensive allocation $X = (i,j)$ where

$$i = ak + b < j = ak + c$$

with a, b, and c being integers and $b < c$. The expected survival values for T_1 and T_2 are

$$S_1(i,k) = (1-rq^{a+1})^b(1-rq^a)^{k-b} \qquad \text{and}$$

$$S_2(j,k) = (1-rq^{a+1})^c(1-rq^a)^{k-c}.$$

When an additional defender is considered, the potential total survival values are

$$\begin{aligned} S(i+1,j) &= S_1(i+1,k) + S_2(j,k) \\ &= (1-rq^{a+1})^{b+1}(1-rq^a)^{k-b-1} + (1-rq^{a+1})^c(1-rq^a)^{k-c}, \end{aligned}$$

and

$$\begin{aligned} S(i,j+1) &= S_1(i,k) + S_2(j+1,k) \\ &= (1-rq^{a+1})^b(1-rq^a)^{k-b} + (1-rq^{a+1})^{c+1}(1-rq^a)^{k-c-1}. \end{aligned}$$

The difference is

$$\Delta S = S(i,j+1) - S(i+1,j)$$

$$= rpq^a(1-rq^a)^{k-c-1}(1-rq^{a+1})^b[(1-rq^{a+1})^{c-b} - (1-rq^a)^{c-b}].$$

This will be positive for all probability values, r and p. Thus when two equal valued targets have the same number (k) of attackers and are defended by numbers of defenders (i and j) which are within the same multiple (a) of the attackers, then additional defenders should be assigned to the target which already has the greatest number of defenders. Note that, for $a - 0$, this includes the situation $i < j < k$ previously considered. This concentration of Defense resources is the converse of the balancing of resources against multiple attackers on a single target described in Appendix A.

XI. ATTACK ON TWO TARGETS.

As in the defense of a target pair, the preferred offensive weapon allocation can be established for attack on two point targets when the defensive allocation $(X = (x_1,x_2))$ and the total attack size (m) is specified. This is outlined in Appendix C for a few limited defensive allocations and attack sizes. Some of the results are summarized in Table 2. This Table, like Table 1, is only partially completed as its purposes are to aid in the search for general offensive strategies and to demonstrate the effect of the success probabilities on the selection of preferred offensive strategies.

Although the specific cases presented in Apprendix C are not very extensive, they do indicate a somewhat more complex structure for the relationship between preferred attacker allocations and success probabilities than was apparent in the corresponding relationship for preferred defender allocations. Tentative conclusions to be drawn are that the attackers should concentrate on one target (the most lightly defended of two equal-valued point targets if there is unequal defense) and to allocate attackers to the other target when the expected damage to the first target is near unity. This is indicated in Table 2 with higher concentrations on the first target being required when the attacker success probability is low (e.g., $r = 0.3$).

The results obtained in Appendix C can also be used to examine the effect of defender success probability on attacker allocation preference. Table 3 contains this information.

TABLE 2

Two-Target Attack Allocation $Y = (y_1, y_2)$ When Defense Is $X = (x_1, x_2)$

$p = 0.9$

efense (x_1,x_2)	Attack size m	r = 0.3 Pref. Y (y_1,y_2)	r = 0.3 Damage $D(y_1,y_2)$	r = 0.5 Y (y_1,y_2)	r = 0.5 Damage D	r = 0.7 Y	r = 0.7 D	r = 0.9 Y	r = 0.9 D	r = 1.0 Y	r = 1.0 D
(1,1)											
	1	(1,0)	.03	(1,0)	.05	(1,0)	.07	(1,0)	.09	(1,0)	.10
	2	(2,0)	.321	(2,0)	.525	(2,0)	.721	(2,0)	.909	(2,0)	1.0
	3	(3,0)	.525	(3,0)	.763	(3,0)	.916	(3,0)	.991	(2,1)	1.1
	4	(4,0)	.667	(2,2)	1.05	(2,2)	1.44	(2,2)	1.82	(2,2)	2.0
	5	(5,0)	.767								
	6										
(1,2											
	1	(1,0)	.03	(1,0)	.05	(1,0)	.07	(1,0)	.09	(1,0)	.10
	2	(2,0)	.321	(2,0)	.525	(2,0)	.721	(2,0)	.909	(2,0)	1.0
	3	(3,0)		(3,0)						(2,1)	
	4									(2,2)	
	5									(2,3)	2.0
	6										
(2,2)											
	1	(1,0)		(1,0)		(1,0)		(1,0)		(1,0)	
	2	(2,0)		(2,0)		(2,0)		(2,0)		(2,0)	
	3	(3,0)		(3,0)		(3,0)		(3,0)		(3,0)	
	4	(4,0)								(3,1)	
	5									(3,2)	
	6									(3,3)	2.0

TABLE 3

Two-Target Attack Allocation Against Specified Defense

r = 0.9

Defense X = (x_1,x_2)	Attack size m	p = 0.3 Y (y_1,y_2)	p = 0.3 Damage D	p = 0.5 Y	p = 0.5 D	p = 0.7 Y	p = 0.7 D	p = 0.9 Y	p = 0.9 D	p = 1.0 Y	p = 1.0 D
(1,1)											
	1	(1,0)	.63	(1,0)	.45	(1,0)	.27	(1,0)	.09	(1,0)	0
	2	(1,1)		(2,0)		(2,0)		(2,0)	.909	(2,0)	.9
	3	(2,1)		(2,1)		(2,1)		(2,1)		(3,0)	
	4	(2,2)		(2,2)		(4,0)		(4,0)		(4,0)	
	5										
	6										
(1,2)											
	1	(1,0)		(1,0)		(1,0)		(1,0)		(1,0)	
	2	(1,1)		(2,0)		(2,0)		(2,0)		(2,0)	
	3	(1,2)		(1,2)		(1,2)		(3,0)		(3,0)	
	4	(2,2)		(2,2)		(2,2)		(2,2)		(2,2)	
	5										
	6										

Derivation of general attack allocation principles is more difficult than the derivation of principles for defensive allocation. This is a consequence of the fact that the addition of a single attacker for one target can require substantial modification of defender assignments for that target. For example, if a target has one attacker and ten defenders and an additional attacker is allocated to it, then the defense is shifted from 10-on-1 to 5-on-1 using the principle presented in Appendix A whereas an additional defender allocated to the target would merely change the defense from 10-on-1 to 11-on-1. Nevertheless, some principles can be established.

Consider a pair of equal-valued targets with defense $X = (k,k+1)$ and let the attack size be $m = 2i$ where $k < i$. If the attack is balanced, $y = (i,i)$, then the expected total damage is

$$D(i,i) = 2 - S_1(k,i) - S_2(k+1,i) =$$
$$2 - (1-rq)^k(1-r)^{i-k} - (1-rq)^{k+1}(1-r)^{i-k-1}$$

If an additional attacker is available and is assigned to the first target (T_1) then the expected total damage is

$$D(i+1,i) = 2 - S_1(k,i+1) - S_2(k+1,i) =$$
$$2 - (1-rq)^k(1-r)^{i+1-k} - (1-rq)^{k+1}(1-r)^{i-k-1}.$$

If the additional attacker is assigned to T_2, then

$$D(i,i+1) = 2 - S_1(k,i) - S_2(k+1,i+1) =$$
$$2 - (1-rq)^k(1-r)^{i-k} - (1-rq)^{k+1}(1-r)^{i-k}.$$

The difference in expected damage is

$$D(i,i+1) - D(i+1,i) = r^2 p(1-rq)^k (1-r)^{i-k-1}$$

$$\geq 0 \quad \text{for all } r,p.$$

Thus the additional attacker should be assigned to T_2 and the attacker allocation $Y = (i,i+1)$ is preferred to $Y = (i+1,i)$.

Suppose another additional attacker is available. For $Y = (i+1,i+1)$ the expected target damage is

$$D(i+1,i+1) = 2 - S_1(k,i+1) - S_2(k+1,i+1)$$

$$2 - (1-rq)^k (1-r)^{i+1-k} - (1-rq)^{k+1} (1-r)^{i-k}$$

and for $Y = (i,i+2)$ it is

$$D(i,i+2) = 2 - S_1(k,i) - S_2(k+1,i+2) =$$

$$2 - (1-rq)^k (1-r)^{i-k} - (1-rq)^{k+1} (1-r)^{i+1-k}.$$

The difference is

$$D(i+1,i+1) - D(i,i+2) = r^2 q(1-rq)^k (1-r)^{i-k}$$

$$\geq 0 \quad \text{for all } r,p.$$

The attacker allocation $Y = (i+1,i+1)$ is preferred to $Y = (i,i+2)$.

The generalization just derived is applicable when the Offense is numerically superior to the Defense. Under this condition a further generalization is possible. Consider a defensive allocation $X = (k,\ell)$ and an offensive allocation $Y = (i,j)$ where $k \leq \ell < i \leq j$. With an additional attacker, the Offense must choose between $Y = (i+1,j)$ and $Y = (i,j+1)$. The expected damage values are

$$D(i+1,j) = 2 = S_1(k,i+1) - S_2(\ell,j) =$$

$$2 - (1-rq)^k(1-r)^{i+1-k} - (1-rq)^\ell(1-r)^{j-\ell}$$

and

$$D(i,j+1) = 2\ S_1(k,i) - S_2(\ell,j+1) =$$

$$2 - (1-rq)^k(1-r)^{i-k} - (1-rq)^\ell(1-r)^{j+1-\ell}.$$

The difference is

$$D(i,j+1) - D(i+1,j) = r(1-rq)^\ell(1-r)^{j-\ell} - r(1-rq)^k(1-r)^{i-k}.$$

Set

$$u = \min\ [(i-k),(j-\ell)],$$

$$w_1 = \begin{cases} (j-\ell) - u & \text{if } (j-\ell) \geq (i-k) \\ 0 & \text{otherwise} \end{cases}, \text{ and}$$

$$w_2 = \begin{cases} (i-k) - u & \text{if } (i-k) > (j-\ell) \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$w_2 - w_1 = (\ell-k) - (j-i).$$

The difference then becomes

$$D(i,j+1) - D(i+1,j) = r(1-rq)^{\ell-k}(1-r)^u[(1-rq)^{\ell-k}(1-r)^{w_1} - (1-r)^{w_2}]$$

$$\geq 0 \qquad \text{when } (1-rq)^{\ell-k} \geq (1-r)^{(\ell-k) - (j-i)}$$

The conditional inequality for $Y(i,j+1)$ to be preferred over $Y(i+1,j)$ can be restated in the form

$$(1-r)^{j-i} \geq \left(\frac{1-r}{1-rq}\right)^{\ell-k}.$$

Note also that

$$1 - r \le 1 - rq$$

so that both sides of the conditional inequality are integral powers of numbers which are less than unity.

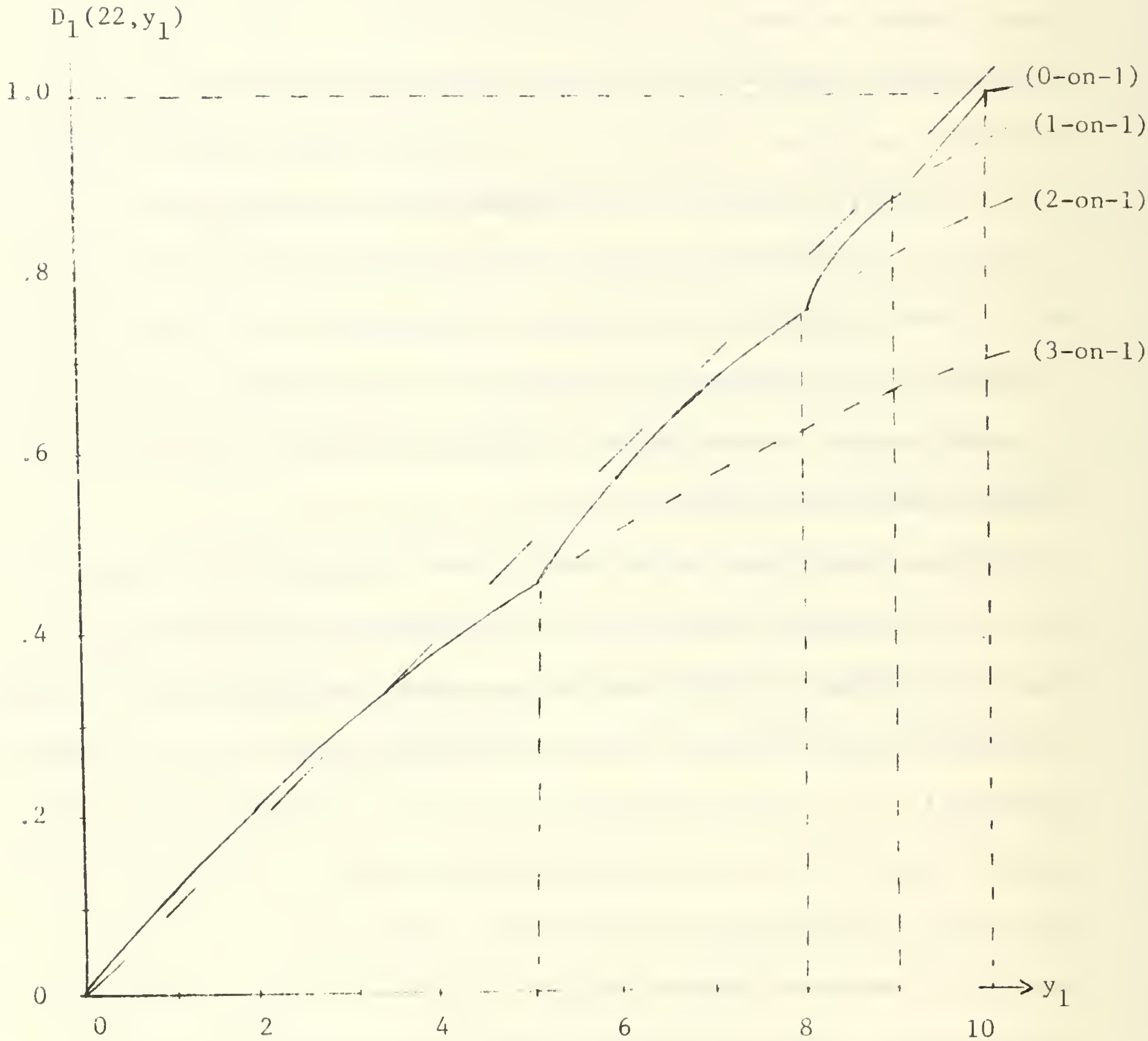
Some general comments on the inequality are in order. When $i = j$ and $k < \ell$ the inequality holds for all r and p and the Offense will prefer $Y = (i,j+1)$ to $Y = (i+1,j)$. Also, if $j - i = \ell - k$ the inequality does not hold and $Y = (i+1,j)$ is preferred to $Y = (i,j+1)$. Since $(1-r)/(1-rq) \le (1-r)$, the exponent $(\ell-k)$ must be larger than $(j-1)$ for the inequality to hold. In general, if the Defense is fairly well balanced $((\ell-k)$ is small) the Offense should tend toward a balanced attack (assign additional attackers to the target with the fewest attackers). The inequality demonstrates the dependence of attacker allocation on the success probabilities r and p as well as the target allocations i,j,k, and ℓ.

XII. DISCUSSION AND CONCLUSIONS.

For a set of point targets, the effect of imperfect weapons on target damage and weapon allocation preferences is quite clear. How much use this theory will be in application to a realistic situation is another question. Either side in a conflict must establish values for the point targets and for the weapon success probabilities r and p. In addition, the opponents weapon allocation must be known. When the Proportional Defense Model can be presumed the Offense needs to know the defensive payoff limitation and the Defense's specification of target values. With this information, the Offense can estimate the Defense's weapon allocation.

In the description of the Proportional Defense Model (Section VI), the average expected damage per attacker (payoff to the Offense) is implicitly assumed to be an appropriate measure of effectiveness. This requirement that the payoff be limited for any attack size is somewhat unrealistic. The Defense should not be overly concerned about small attack sizes for which the total expected damage is not great even though the payoff exceeds the prescribed limit. This can be illustrated for Example 2. When the Defense assigns only three defenders against the first attacker and leaves the remainder of the defense of target T_1 unchanged, the situation will be as shown in Figure 8. If an expected damage of 0.3009 or less is not catastrophic to the overall defense of the set of targets or if it can be assumed that the Offense will not consider a damage level this low as acceptable, then the Defense can eliminate the additional defender and still achieve a Proportional Defense with the same payoff limitation against larger size attacks.

$v_1 = 1 \quad r - 0.9 \quad p = 0.5 \quad d^* = 0.1$



QUALIFIED PROPORTIONAL DEFENSE OF T_1

FIGURE 8

The restriction of payoff limitation to attacks of size large enough to produce damage greater than some specified damage level does not conform to the Proportional Defense Model described in Section VI but is consistent with the principle of Proportional Defense as described in Section V. For Example 2, both the Proportional Defense Model solution and its modification by deletion of a defender specify an attack by nine attackers as exhaustive and limit the payoff to the Offense to $d^* = 0.1$ for attacks of this size or larger. The strict Proportional Defense Model has this limitation on payoff for all attack sizes whereas with the modification this payoff limitation holds only for attack sizes of four or larger with expected damage greater than 0.3009. If the Defense can determine a damage level D^* such that the payoff limitation is required only for attack sizes which produce damage levels exceeding D^*, then some reduction in the Defenses resources from the requirements for the Proportional Defense Model will be possible. The proposed modification can be called a Qualified Proportional Defense Model with the qualification consisting of specification of the minimum damage level D^*, for which the payoff limitation will be required.

Reservations about the usefulness of the model described in this report pertain principally to the assumptions used in establishing the model. In addition to the assumptions which eliminate interactions (e.g., no interference between defenders of a given target) the model is restricted to point targets. Considerable extension or reformulation of the model is required to make it applicable to area targets. For this important class of targets, a single attacker cannot, in general, destroy the entire target area and the simple dichotomy that the target is either

completely destroyed or undamaged by the attacker is no longer appropriate. Thus, there is a dependence of expected damage by any one attacker on the expected damage by preceding attackers of the same target which must be considered.

A substantial development of damage estimates for area targets using statistical distributions is available in the literature [1]. Attacker accuracy and lethal range are considerably but attacker reliability is not included in damage estimation procedure. In extension of the model for point targets to area targets, it appears to be essential to separate these contributors to the attacker success probability which lead to a substantially more complicated model.

In summary, it can be asserted that, as would be expected, weapon success probabilities have a considerable effect on target damage estimates and hence, on preferred weapon assignments by both Defense and Offense. As a general principle the Defense should distribute its defenders of a given target against the attackers of that target as uniformly as possible regardless of the weapon success probabilities. The Defense should, however, concentrate defender allocations to individual targets to maximize expected target survival values with the extent of the concentration being highly dependent upon the weapon success probabilities. General principles for weapon allocation by the Offense are more difficult to establish. Limited investigation does, however, indicate that the Offense should tend toward a balanced attack on targets of equal value when it has more weapons than the Defense.

APPENDIX A

Target Defense Against An Attack Wave

Suppose the Defense has x_{ij} defenders to be used against y_{ij} attackers in the $j^{\underline{th}}$ attack wave on T_i. The defenders are to be distributed against the attackers so as to maximize the expected target survival value

$$V_{ij}(x_{ij},y_{ij}) = V_{i,j-1}P(x_{ij},y_{ij})$$

where $V_{i,j-1}$ is the expected survival value of the target after the $(j-1)^{st}$ attack wave. This is accomplished when $P(x_{ij},y_{ij})$ is maximized by the appropriate defender allocation.

If $x_{ij} \leq y_{ij}$ and a 1-on-1 defense is used, then the probability that the target survives the $j^{\underline{th}}$ attack wave is

$$P_1(x_{ij},y_{ij}) = (1-rq)^{x_{ij}}(1-r)^{y_{ij}-x_{ij}}.$$

If a single defender is shifted from one attacker to provide a 2-on-1 defense against one attacker, then the survival probability becomes

$$P_2(x_{ij},y_{ij}) = (1-rq^2)(1-rq)^{x_{ij}-2}(1-r)^{y_{ij}-x_{ij}+1}.$$

The difference in survival probabilities is

$$P_1(x_{ij},y_{ij}) - P_2(x_{ij},y_{ij}) = (1-rq)^{x_{ij}-2}(1-r)^{y_{ij}-x_{ij}}r(1-q)^2.$$

and is positive for all probabilities r and q. Additional shifts will produce a similar result. A 1-on-1 defense is preferred in this case.

If $$x_{ij} > y_{ij}, \quad \text{let}$$

$$x_{ij} = ky_{ij} + \ell$$

where k is an integer greater than zero and ℓ is an integer less than Y_{ij}. When the defenders are distributed as evenly as possible ($k+1$ defenders against each of ℓ attackers and k defenders against each of the remaining $y_{ij} - \ell$ attackers) the probability that the target survives the $j^{\underline{th}}$ attack wave is

$$P_1(x_{ij},y_{ij}) = (1-rq^{k+1})^{\ell}(1-rq^k)^{y_{ij}-\ell}.$$

If a defender is shifted from an attacker with a k-on-1 defense to an attacker with a $(k+1)$-on-1 defense, then the probability that the target survives is

$$P_2(x_{ij},y_{ij}) = (1-rq^{k+2})(1-rq^{k+1})^{\ell-1}(1-rq^k)^{y_{ij}-\ell-1}(1-rq^{k-1}).$$

The difference in target survival values is

$$P_1(x_{ij},y_{ij}) - P_2(x_{ij},y_{ij}) = (1-rq^{k+1})^{\ell-1}(1-r)^{y_{ij}-\ell-1}rq^{k-1}(1-q)(1-q^2).$$

This quantity is positive for all probabilities r and q. Attempts to shift defenders further from this balanced defender allocation produce similar results also.

From the above analysis it can be seen that the preferred defender allocation against an attack wave distributes the defenders as evenly as possible against the attackers (a balanced defense).

APPENDIX B

Preferred Defense Against Attack $Y = (y_1, y_2)$

$Y = (1,1)$

Compare Defense Allocations $X_1 = (i, n-i)$ and $X_2 = (i+1, n-i-1)$

$$S(i,n-i) = S_1(i,1) + S_2(n-1,1) = (1-rq^i) + (1-rq^{n-i})$$

$$S(i+1,n-i-1) = S_1(i+1,1) + S_2(n-i-1) = (1-rq^{i+1}) + (1-rq^{n-i-1})$$

$$S(i,n-1) - S(i+1,n-i-1) = r(1-q)(q^{n-i-1}-q^i)$$

$$\geq 0 \qquad \text{when } i \geq \frac{n-1}{2} .$$

n No. Defenders	(x_1, x_2) Preferred	$S(i,j)$ Expected Survival
1	(1,0)	$S(1,0) = (1-rq) + (1-r)$
2	(1,1)	$S(1,1) = 2(1-rq)$
3	(2,1)	$S(2,1) = (1-rq^2) + (1-rq)$
4	(2,2)	$S(2,2) = 2(1-rq^2)$
5	(3,2)	$S(3,2) = (1-rq^3) + (1-rq^2)$
6	(3,3)	$S(3,3) = 2(1-rq^3)$

$Y = (1,2)$

<u>n = 1</u>

Total Survival Values

$$S(1,0) = S_1(1,1) + S_2(0,2) = (1-rq) + (1-r)^2$$

$$S(0,1) = S_1(0,1) + S_2(1,2) = (1-r) + (1-rq)(1-r)$$

Difference in Survival Values

$$S(1,0) - S(0,1) = r^2 p \geq 0 \quad \text{for all } r,p$$

Preferred Defense Allocation $X = (1,0)$

n = 2

Total Survival Values

$$S(2,0) = (1-rq^2) + (1-r^2), \quad S(1,1) = (1-rq) + (1-rq)(1-r)$$

$$S(0,2) = (1-r) + (1-rq)^2$$

Differences in Survival Values

$$S(1,1) - S(0,2) = r^2pq \geq 0 \quad \text{for all } r,p.$$

$$S(2,0) - S(1,1) = rp(r-p)$$

Preferred Defense Allocation

$$X = \begin{cases} (1,1) & \text{when } r \geq p \\ (2,0) & \text{when } r < p \end{cases}$$

n = 3

Total Survival Values

$$S(3,0) = (1-rq^3) + (1-r)^2, \quad S(2,1) = (1-rq^2) + (1-rq)(1-r)$$

$$S(1,2) = (1-rq) + (1-rq^2), \quad S(0,3) = (1-r) + (1-rq^2)(1-rq)$$

Differences in Survival Values

$$S(1,2) - S(0,3) = rp(p+rq^2) \geq 0 \quad \text{for all } r,p$$

$$S(3,0) - S(2,1) = rp(r-2p+p^2) \geq 0 \quad \text{when } r \geq 2p - p^2 = r^*$$

$$S(2,1) - S(1,2) = rp(r-rp-p) \geq 0 \quad \text{when } r \geq \frac{p}{1-p} = r^{**}$$

$$S(3,0) - S(1,2) = rp(2r-rp+p^2-3p) \geq 0 \quad \text{when } r \geq \frac{3p-p^2}{2-p} = r^{***}$$

(Note that $r^* = r^{**} = r^{***} = 0.618$ when $p = 0.382$)

Preferred Defense Allocation

if $p < 0.382$

$$X = \begin{cases} (1,2) & \text{when } r < r^{**} \\ (2,1) & \text{when } r^{**} \leq r < r^* \\ (3,0) & \text{when } r^* \leq r \end{cases}$$

if ≥ 0.382

$$X = \begin{cases} (1,2) & \text{when } r < r^{***} \\ (3,0) & \text{when } r \geq r^{***} \end{cases}$$

$n = 4$

Total Survival values $S(4,0)$, $S(3,1)$, $S(2,2)$, $S(1,3)$, $S(0,4)$

Differences in Survival Values

$$S(1,3) - S(0,4) = rp(p+rq^2) \geq 0 \quad \text{for all } r,p$$

$$S(2,2) - S(1,3) = r^2pq^2 \geq 0 \quad \text{for all } r,p$$

$$S(4,0) - S(3,1) = rp(r-1+q^3) \geq 0 \quad \text{when } r \geq 1-q^3 = r^*$$

$$S(3,1) - S(2,2) = rp(rq-1+q^2) \geq 0 \quad \text{when } r \geq \frac{1-q^2}{q} = r^{**}$$

$$S(4,0) - S(2,2) = rp[r(2-p) - p(5-4p+p^2)] \geq 0 \quad \text{when } r \geq \frac{p(5-4p+p^2)}{2-p} = r^{***}$$

($r^* = r^{**} = r^{***} = 0.576$ when $p = 0.248$)

Preferred Defense Allocation

if $p < 0.248$

$$X = \begin{cases} (2,2) & \text{when } r < r^{**} \\ (3,1) & \text{when } r^{**} < r < r^* \\ (4,0) & \text{when } r^* < r \end{cases}$$

if $p \geq 0.248$

$$X = \begin{cases} (2,2) & \text{when } r < r^{***} \\ (4,0) & \text{when } r^{***} \leq r \end{cases}$$

$Y = (2,2)$

<u>$n = 2$</u>

Total Survival Values

$$S(2,0) = (1-rq)^2 + (1-r)^2, \quad S(1,1) = 2(1-rq)(1-r)$$

Difference

$$S(2,0) - S(1,1) = rp^2 \geq 0 \quad \text{for all } r,p$$

Preferred Defense Allocation

$$X = (2,0)$$

<u>$n = 3$</u>

Total Survival Values

$$S(3,0) = (1-rq^2)(1-rq) + (1-r)^2, \quad S(2,1) = (1-rq)^2 + (1-rq)(1-r)$$

Difference

$$S(3,0) - S(2,1) = rp[r(1-q^2) - p] \geq 0 \quad \text{when } r \geq \frac{p}{1-q^2} = r^*$$

Preferred Defense Allocation

$$X = \begin{cases} (3,0) & \text{when } r \geq r^* \\ (2,1) & \text{when } r < r^* \end{cases}$$

<u>$n = 4$</u>

Total Survival Values

$$S(4,0) = 1-rq^2)^2 + (1-r)^2, \quad S(3,1) = (1-rq^2)(1-rq) + (1-rq)(1-r)$$

$$S(2,2) = 2(1-rq)^2$$

Differences

$$S(2,2) - S(3,1) = rp^2(1-rq) \geq 0 \quad \text{for all } r,p$$

$$S(4,0) - S(2,2) = r[r(1-q^2)^2 - 2p^2] \geq 0 \quad \text{when } r \geq \frac{2p^2}{(1-q^2)^2} = r^*$$

Preferred Defense Allocation

$$X = \begin{cases} (4,0) & \text{when} \quad r \geq r^* \\ (2,2) & \text{when} \quad r < r^* \end{cases}$$

APPENDIX C

Preferred Attack Against Defense $X = (x_1, x_2)$

Total expected damage when attack is $Y = (i,j)$ with $i + j = m$

$$D(i,j) = 2 - S_1(x_1, i - S_2(x_2, j)$$

$X = (1,1)$

<u>$m = 2$</u>

$$D(2,0) = r(1+q-rq), \quad D(1,1) = 2rq$$

$$D(2,0) - D(1,1) = r[p(1+r)-r] \geq 0 \quad \text{when } p \geq \frac{r}{1+r} = p^*$$

Preferred Attack Allocation

$$Y = \begin{cases} (2,0) & \text{when } p \geq p^* \\ (1,1) & \text{when } p < p^* \end{cases}$$

<u>$m = 3$</u>

$$D(3,0) = 1 - (1-rq)(1-r)^2, \quad D(2,1) = r(1+2q-rq)$$

$$D(3,0) - D(2,1) = r[p(1+r-r^2) - r(2-r)] \geq 0 \quad \text{when} \quad p \geq \frac{r(2-r)}{1+r-r^2} = p^*$$

Preferred Attack Allocation

$$Y = \begin{cases} (3,0) & \text{when } p \geq p^* \\ (2,1) & \text{when } p < p^* \end{cases}$$

<u>$m = 4$</u>

$$D(4,0) = 1 - (1-rq)(1-r)^3, \quad D(3,1) = 2 - (1-rq)(1-r)^2 - (1-rq)$$

$$D(2,2) = 2 - 2(1-rq)(1-r)$$

$$D(2,2) - D(3,1) = r^2(1-rq) \geq 0 \quad \text{for all } r,p$$

$$D(4,0) - D(2,2) = r[p(1-r)(1+2r-r^2) - r(2-r)^2]$$

$$\geq 0 \quad \text{when } p \geq \frac{r(2-r)^2}{(1-r)(1+2r-r^2)} = p^*$$

Preferred Attack Allocation

$$Y = \begin{cases} (4,0) & \text{when } p \geq p^* \\ (2,2) & \text{when } p < p^* \end{cases}$$

$X = (1,2)$

<u>$m = 2$</u>

$$D(2,0) = r(1+q-rq), \quad D(1,1) = rq(1+q), \quad D(0,2) = rq(2-rq)$$

$$D(2,0) - D(0,2) = rp(1-rq) \geq 0 \quad \text{for all } r,p$$

$$D(2,0) - D(1,1) = r[(1-q^2) - rq]$$

$$\geq 0 \quad \text{when } r \leq \frac{1-q^2}{q} = r^*$$

Preferred Attack Allocation

$$Y = \begin{cases} (2,0) & \text{when } r \leq r^* \\ (1,1) & \text{when } r > r^* \end{cases}$$

<u>$m = 3$</u>

$$D(3,0) = 1 - (1-rq)(1-r)^2, \quad D(2,1) = 2 - (1-rq)(1-r) - (1-rq^2)$$

$$D(1,2) = 2 - (1-rq) - (1-rq)^2, \quad D(0,3) = 1 - (1-rq)^2(1-r)$$

$$D(3,0) - D(0,3) = rp(1-r)(1-rq) \geq 0 \quad \text{for all } r,p$$

$$D(3,0) - D(2,1) = r[(1-rq)(1-r)-q^2] = rf_1$$

$$D(3,0) - D(1,2) = (1-rq)[1 + (1-rq)(1-r)^2] - 1 = f_2$$

$$D(2,1) - D(1,2) = rp[p(1-r) - r] = rpf_3$$

Preferred Attack Allocation (Figure 9)

$$= \begin{cases} (3,0) & \text{when } f_1 \geq 0,\ f_2 \geq 0 \\ (2,1) & \text{when } f_1 < 0,\ f_3 \geq 0 \\ (1,2) & \text{when } f_2 < 0,\ f_3 < 0 \end{cases}$$

<u>m = 4</u>

$$D(4,0) = 1 - (1-rq)(1-r)^3, \quad D(3,1) = 2 - (1-rq)(1-r)^2 - (1-rq^2)$$

$$D(2,2) = 2 - (1-rq)(1-r) - (1-rq)^2, \quad D(1,3) = 2 - (1-rq) - (1-rq)^2(1-r)$$

$$D(0,4) = 1 - (1-rq)^2(1-r)^2$$

$$D(2,2) - D(1,3) = r^2q(1-rq) \geq 0 \quad \text{for all } r,p$$

$$D(4,0) - D(0,4) = rp(1-rq)(1-r)^2 \geq 0 \quad \text{for all } r,p$$

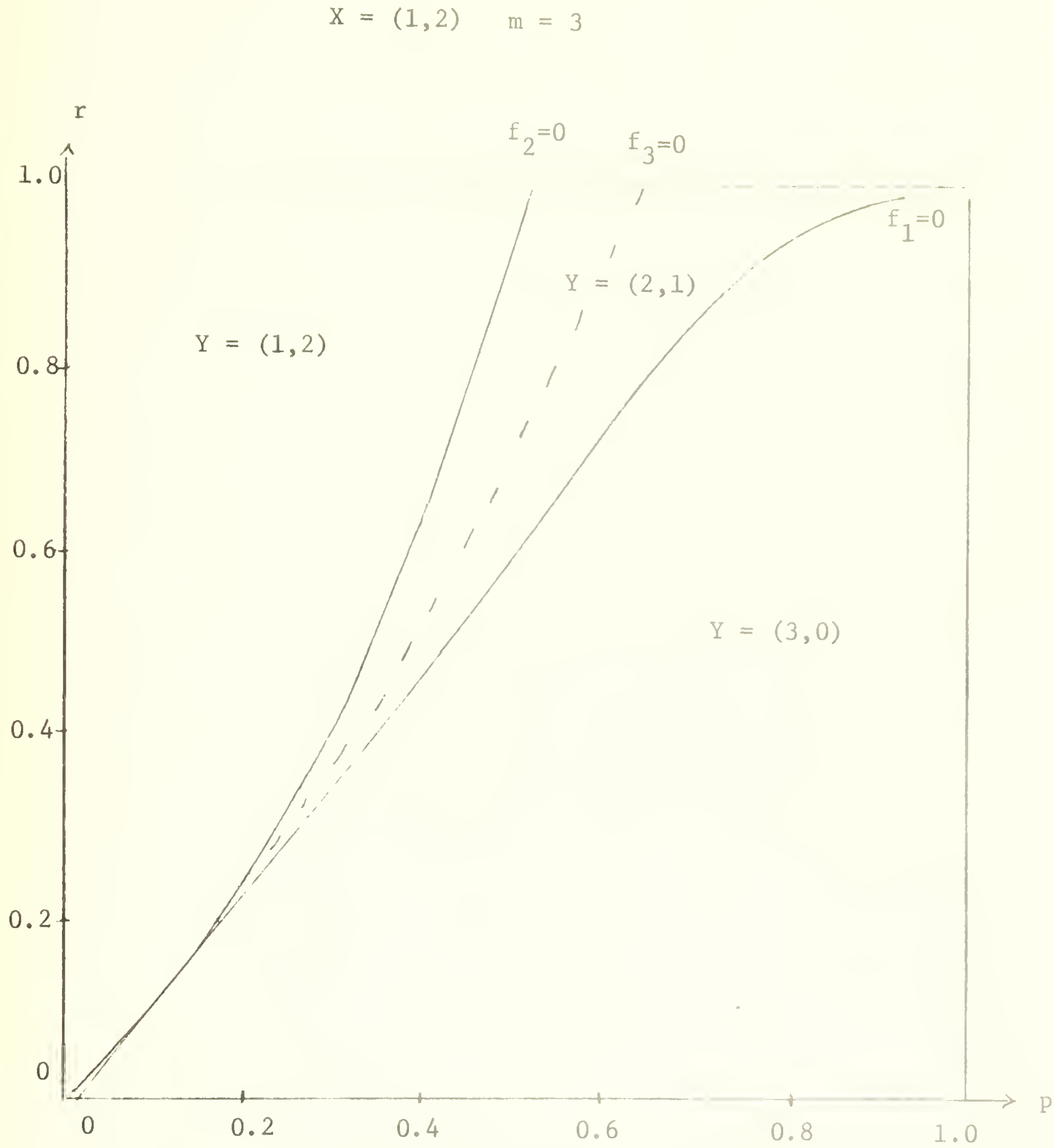
$$D(4,0) - D(3,1) = r[(1-rq)(1-r)^2 - q^2] = rf_1$$

$$D(3,1) - D(2,2) = r(1-r)(1-rq) + (1-rq)^2 - (1-rq^2) = f_2$$

$$D(4,0) - D(2,2) = r[(1-r)(2-r)(1-rq) - q(2-rq)] = rf_3$$

Preferred Attack Allocation

$$= \begin{cases} (4,0) & \text{when } f_1 \geq 0,\ f_3 \geq 0 \\ (3,1) & \text{when } f_1 < 0,\ f_2 \geq 0 \\ (2,2) & \text{when } f_3 < 0,\ f_2 < 0 \end{cases}$$



Preferred Attack Allocation $Y = (y_1, y_2)$

Figure 9

REFERENCES

[1] Eckler, A. R. and Burr, S. A. "Mathematical Models of Target Coverage and Missile Allocation, Military Operations Research Society, 1972.

[2] Howard, G. T. and Shumate, K. C. "A Proportional Defense Model," Naval Research Logistic Quarterly, March 74, V21, #1.

[3] Matlin, S., "A Review of the Literature on the Missile Allocation Problem," Operations Research 18, (1970) pp. 334-373.

DISTRIBUTION LIST

Dr. Bill Mitchell
Department of Management Sciences
School of Business and Economics
California State University
Hayward, California 94542

Technical Library
Naval Ordnance Station
Indian Head, Maryland 20640

Mr. W. L. Nicholson
Staff Scientist
Pacific Northwest Laboratories
Battelle Boulevard
Richland, Washington 99352

D. R. Barr 1
Code 55Bn
Naval Postgraduate School
Monterey, California 93940

D. P. Gaver 1
Code 55Gv
Naval Postgraduate School
Monterey, California 93940

G. T. Howard 1
Code 55Hk
Naval Postgraduate School
Monterey, California 93940

J. G. Taylor 1
Code55Tw
Naval Postgraduate School
Monterey, California

J. Bryce Tysver 10
Code 55Ty
Naval Postgraduate School
Monterey, California 93940

Defense Documentation Center Cameron Station Alexandria, Virginia 22314	12
Dean of Research Naval Postgrauduate School Monterey, California 93940	1
Library Code 0212 Naval Postgraduate School Monterey, California 93940	2
Library, Code 55 Department of Operations Research and Administrative Sciences Naval Postgraduate School Monterey, California 93940	3
CDR Angelo Ciccolani Strategic Systems Project Office Crystal Mall Washington, D. C. 20390	6
Naval Research Laboratory Washington, D. C. 20390	1
Naval Ship Research and Development Center Washington, D. C. 20034	1
Naval Undersea Research and Development Center San Diego, California 92132	1
Robert Miller Office of Naval Research Arlington, Virginia 22217	1
Thomas Varley Office of Naval Research Arlington, Virginia 22217	1
Naval Weapons Center China Lake, California 93555	1

B. Buc 1
Operations Research, Inc.
1400 Spring Street
Silver Spring, Maryland 20910

Applied Physics Laboratory 1
John Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland 20910

U [redacted]